Model Reduction and the Design of Reduced-Order Control Laws

This paper is concerned with the problem of designing satisfactory low-order (incomplete state feedback) controllers starting with a high-order, state-space model. Two design approaches are considered: the control law reduction technique, recently developed by the authors (Wilson et al., 1973), and the well-known model reduction approach. These two design techniques are used to develop a variety of low-order controllers for a double-effect evaporator starting with a 10th-order model. Experimental and simulated response data from the computer-controlled evaporator demonstrate the superiority of the control law reduction approach in this application. It is also shown that several of the previously published modal approaches to model reduction are basically equivalent since they yield identical reduced-order models.

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SCOPE

In many practical applications of modern control theory the most difficult problem is to obtain a suitable process model. An analytical approach using basic chemical engineering principles often results in a dynamic model which consists of a large number of nonlinear differential equations. The model is usually too complicated for use in controller design or for implementation as part of the actual control system. Consequently, for purposes of control system design it is common practice to linearize the model and assume time-invariant behavior. Fortunately, the resulting linear "state-space model" (that is, a set of first-order differential equations) will usually adequately describe the process transients in the region of normal operation and provide a suitable basis for control system design. However, many of the multivariable control design techniques (Gould, 1969) result in a control law that requires the availability of all elements of the state vector. Such control laws are frequently impractical because of their complexity and because in many applications it is not practical to measure or estimate all of the state variables. Hence there has been widespread interest in developing control laws that require only a subset of the state vector to be available. The resulting controllers are usually referred to as *incomplete state feedback* or *low-order controllers*.

This investigation is concerned with the problem of designing a satisfactory low-order, multivariable controller starting with a high-order, state-space model of a process. To be judged satisfactory, the low-order controller should perform almost as well as more complicated high-order (state-feedback) controllers. Emphasis is placed on discrete process models and discrete controllers since they are more convenient than their continuous counterparts for on-line implementation via digital process computers. Two basic approaches were used to design low-order controllers:

1. Model Reduction Approach. The original high-order model is simplified using a modal analysis to eliminate selected state variables, that is, to reduce the order of the model. The resulting low-order model then serves as a basis for designing a low-order controller using standard state-feedback design methods (for example, optimal control).
2. Control Law Reduction Approach. The high-order model is first used to design a high-order controller using standard state-feedback design techniques. The high-order controller is then simplified by eliminating selected state variables via a modal analysis of the high-order model. A low-order controller results. In both approaches, the high-order and low-order controllers can include integral feedback, feedforward and setpoint terms, in addition to proportional feedback.

The first approach, model reduction, has been applied in many previous investigations (for example, Nicholson, 1964, 1967; Anderson, 1969). The second approach, control law reduction, has recently been developed by the present authors (Wilson et al., 1973; Wilson, 1974). An objective of this investigation is to critically evaluate the performance of low-order, multivariable controllers designed using these two approaches. The evaluation includes both experimental and simulated data for a double effect evaporator.

CONCLUSIONS AND SIGNIFICANCE

The two design approaches mentioned above were used to design reduced-order controllers for a computer-controlled, double effect evaporator at the University of Alberta. The starting point in the design was a theoretical 10th-order, state-space model of the evaporator derived from linearized material and energy balances. A variety of model reduction procedures were then employed to generate 3rd, 4th, and 5th-order models as well as 3rd and 5th-order controllers. Additional low-order controllers were designed via the control law reduction approach.

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These low-order controllers were then evaluated in simulation and experimental studies.

The evaluation of the low-order controllers included the following items: several model reduction techniques; different sequences of design steps to arrive at a low-order controller of specified order (see Figure 1); various multivariable controllers which included proportional feedback, integral feedback, feedforward and setpoint control modes; transient responses to a variety of disturbances.

The control law reduction approach developed by the authors (Wilson et al., 1973; Wilson, 1974) produced low-order controllers that were more reliable, smoother-acting and more practical than those resulting from the model reduction approach. Individual elements in the controller matrices produced by the two approaches differed by as much as a factor of three. The controller performance also depended on the modes selected, for example, proportional plus integral control vs. proportional feedback plus feedforward control.

In applying the model reduction approach, it was possible to design satisfactory 3rd-order controllers even though the 3rd-order process models showed significant errors in their open-loop responses. The resulting controllers were practical, easy to implement, and performed almost as well as 5th-order controllers. Furthermore, the 3rd-order controllers were superior to a conventional multi-loop control scheme consisting of three single variable controllers.

Of the various model reduction techniques, those techniques that guarantee agreement between the high-order and the reduced-order models at both the initial conditions and at steady state appear more desirable for control applications. Also the choice of which eigenvalues of the high-order model to retain in the low-order model can be a very important decision (compare Figure 3).

A further conclusion of this study is that several of the previously published modal approaches to model reduction are essentially equivalent since they produce identical reduced-order models. Apparently, the equivalence of these independently developed techniques has not been previously reported.

The design methods and practical experience resulting from this investigation should be of interest to those planning industrial applications of multivariable control techniques.

Multivariable control systems designed using state-space models often require that all of the system state variables are available. However, in most process control problems this is not the actual situation, and consequently there has been considerable interest in the development of control algorithms which require the availability of only some, rather than all, state variables. Such controllers are referred to in this paper as *low-order* or *reduced-order* controllers.

Figure 1 illustrates the various approaches that can be used to design a discrete reduced-order controller starting from a high-order, continuous model. In the model reduction approach, a low-order model is first derived from the high-order model and a controller is then designed for the low-order model using state feedback design methods (for example, Anderson, 1969; Nicholson, 1964, 1967). This controller can then be used as a reduced-order controller for the high-order system. This approach consists of steps 1, 3, and 7 or steps 2, 4, and 7 depending on whether the model reduction step precedes the discretization step or vice versa. Rogers and Sworder (1971) have suggested that the model reduction and controller design steps be performed simultaneously, subject to the additional restriction that the resulting reduced-order controller be the best suboptimal controller for the high-order system.

An alternative approach is to first design a satisfactory state feedback controller for the high-order model and then use it and the high-order model to design the low-order controller (that is, steps 2, 5, and 8 in Figure 1). This approach, control law reduction, has been developed by the authors (Wilson et al., 1973) and does not require the calculation of a low-order model.

A third design approach is to generate a reduced-order controller directly from the high-order model without calculating a reduced-order model or a high-order controller

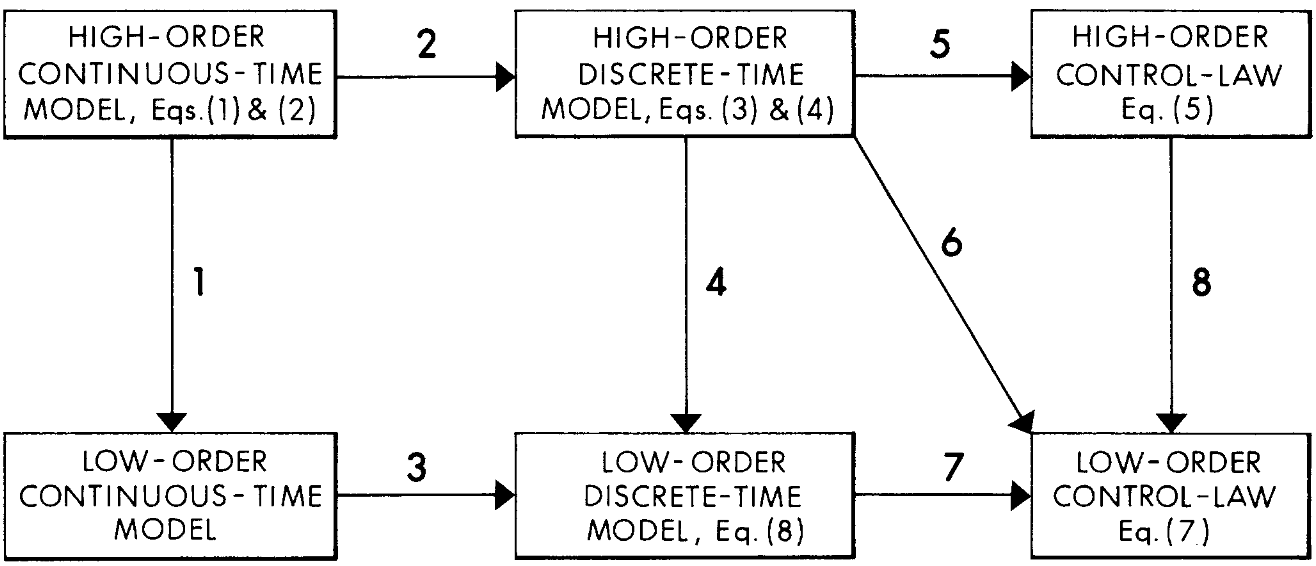

Fig. 1. Diagrammatic representation of the different steps and paths for proceeding from a high-order continuous process model to a low-order discrete control law.

(that is, steps 2 and 6). This approach includes a wide variety of design methods such as modal control (Gould, 1969), specific optimal control, and eigenvalue assignment techniques (Davison, 1972). These design methods are the subject of current investigations by the authors and coworkers and will not be considered further in this paper.

PROBLEM FORMULATION

Consider the linear, time-invariant, continuous-time, state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{D}\,\mathbf{d}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \tag{2}$$

where the state vector $\mathbf{x}$, input vector $\mathbf{u}$, disturbance vector $\mathbf{d}$, and output vector $\mathbf{y}$ are column vectors of dimensions n, m, q and p, respectively. Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices of the appropriate dimensions. The corresponding discrete-time model can be derived from Equations (1) and (2) using well-known techniques (for example, Lapidus and Luus, 1967; Ogata, 1967) and is written

$$\mathbf{x}(j+1) = \boldsymbol{\varphi}\,\mathbf{x}(j) + \boldsymbol{\Delta}\,\mathbf{u}(j) + \boldsymbol{\theta}\,\mathbf{d}(j) \tag{3}$$

$$\mathbf{y}(j) = \mathbf{C}\,\mathbf{x}(j) \quad j = 0, 1, 2, \ldots \tag{4}$$

A general multivariable controller (Newell et al., 1972a, c) for the system in Equations (3) and (4) is given by Equation (5)

$$\mathbf{u}(j) = \mathbf{K}_1^{FB}\,\mathbf{x}_1(j) + \mathbf{K}_2^{FB}\,\mathbf{x}_2(j) + \mathbf{K}^{FF}\,\mathbf{d}(j) + \mathbf{K}^{SP}\,\mathbf{y}^{SP}(j) + \mathbf{K}^{I}\sum_{i=0}^{j}[\mathbf{y}(i) - \mathbf{y}^{SP}(i)] \tag{5}$$

where $\mathbf{x}$ has been partitioned into a l vector $\mathbf{x}_1$ of variables which are available for purposes of control and a $(n-l)$ vector $\mathbf{x}_2$:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{6}$$

Vector $\mathbf{y}^{SP}$ denotes the setpoint of $\mathbf{y}$ and the superscripts on the controller gain matrices (for example, $\mathbf{K}^{FB}$) denote proportional feedback, feedforward, setpoint, and integral feedback control modes, respectively.

The multivariable control law in Equation (5) can only be implemented if a measurement or estimate of each state variable is available. A suitable reduced-order control law which requires only $\mathbf{x}_1$ rather than $\mathbf{x}$ is of the form:

$$\mathbf{u}(j) = \mathbf{K}_R^{FB}\,\mathbf{x}_1(j) + \mathbf{K}_R^{FF}\,\mathbf{d}(j) + \mathbf{K}_R^{SP}\,\mathbf{y}^{SP}(j) + \mathbf{K}_R^{I}\sum_{i=0}^{j}[\mathbf{y}(i) - \mathbf{y}^{SP}(i)] \tag{7}$$

where the subscript R refers to the reduced-order controller. Note that $\mathbf{y}$ is retained in the reduced-order control law since its elements are, by definition, the measured output variables.

The control problem of interest is to design a reduced-order controller of the form of Equation (7) starting with the high-order model in Equations (3) and (4). In the following sections two general design approaches, model reduction and control law reduction, are considered.

MODEL REDUCTION

One approach for designing a reduced-order controller is to first use a model reduction technique to derive a reduced-order model which contains $\mathbf{x}_1$ but not $\mathbf{x}_2$. The resulting reduced-order model and state feedback design methods can then be used to design a reduced-order controller of the form of Equation (7) (Nicholson, 1964, 1967; Anderson, 1969). In the model reduction step, the objective is to derive the reduced-order model in Equation (8) starting from the high-order model in Equation (3):

$$\mathbf{x}_1(j+1) = \boldsymbol{\varphi}_R\,\mathbf{x}_1(j) + \boldsymbol{\Delta}_R\,\mathbf{u}(j) + \boldsymbol{\theta}_R\,\mathbf{d}(j) \tag{8}$$

Existing techniques for reducing the order of a multi-input, multi-output, state-space model can be divided into two categories: modal approaches and least squares approaches (Sinha and De Bruin, 1973; Wilson, 1974). Various modal approaches are briefly described below; least squares methods have been considered elsewhere (Wilson, 1974).

Modal Approach to Model Reduction

In this approach the strategy is to retain certain modes (or eigenvalues) of the high-order model in the low-order model. Consequently, the approach is based on the following modal analysis (for example, Gould, 1969; Friedly, 1972) of the high-order model.

For any square matrix $\boldsymbol{\varphi}$, there exists an $n \times n$ matrix $\mathbf{M}$ which transforms Equation (3) into its Jordan canonical form (Gantmacher, 1959; Friedly, 1972):

$$\begin{bmatrix} \mathbf{z}_1(j+1) \\ \mathbf{z}_2(j+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1 & 0 \\ 0 & \boldsymbol{\alpha}_2 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(j) \\ \mathbf{z}_2(j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{bmatrix} \mathbf{u}(j) + \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} \mathbf{d}(j) \tag{9}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\delta}$ and $\boldsymbol{\eta}$ are defined as

$$\boldsymbol{\alpha} \equiv \mathbf{M}^{-1}\,\boldsymbol{\varphi}\,\mathbf{M} = \begin{bmatrix} \boldsymbol{\alpha}_1 & 0 \\ 0 & \boldsymbol{\alpha}_2 \end{bmatrix} \tag{10}$$

$$\boldsymbol{\delta} \equiv \mathbf{M}^{-1}\,\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{bmatrix}, \quad \boldsymbol{\eta} \equiv \mathbf{M}^{-1}\,\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} \tag{11}$$

and $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are block diagonal matrices with the eigenvalues of $\boldsymbol{\varphi}$ as the diagonal elements. Vector $\mathbf{z}$ is the canonical state vector defined by the similarity transformation

$$\mathbf{x} = \mathbf{M}\,\mathbf{z} \tag{12}$$

or in partitioned form as

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \tag{13}$$

The partitions of $\mathbf{z}$, namely $\mathbf{z}_1$ and $\mathbf{z}_2$, have the same dimensions as $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively.

At this point in the analysis, various expressions for $\boldsymbol{\varphi}_R$, $\boldsymbol{\Delta}_R$, and $\boldsymbol{\theta}_R$ can be derived depending on the particular assumptions that are made (Marshall, 1966; Davison, 1968a, b; Chidambara and Davison, 1967a, b, c). In the original papers, the model reduction techniques were developed for continuous models; however, the extensions to discrete models are straightforward and have been presented by the authors (Wilson et al., 1972; Wilson, 1974). Also, when a modal approach is used to reduce a high-order continuous model to a low-order discrete model, the same low-order model is obtained regardless whether the continuous model is first discretized and then reduced, or vice versa (Wilson et al., 1972).

Marshall's Method

In Marshall's model reduction technique (Marshall, 1966), it is assumed that the dominant eigenvalues of the high-order system are located in $\boldsymbol{\alpha}_1$ and the less significant eigenvalues are located in $\boldsymbol{\alpha}_2$. (The dominant eigenvalues

TABLE 1. DISCRETE REDUCED-ORDER MODELS DERIVED FROM THREE MODAL APPROACHES

Design method	φ_R	Δ_R	θ_R	E_R	F_R
Marshall	$\varphi_1 - \varphi_2 V_4^{-1} V_3$	$\Delta_1 + \varphi_2 V_4^{-1} (I - \alpha_2)^{-1} \delta_2$	$\theta_1 + \varphi_2 V_4^{-1} (I - \alpha_2)^{-1} \eta_2$	—	—
Davison	same	$M_1 \delta_1$	$M_1 \eta_1$	—	—
Revised Davison	same	$M_1 \delta_1$	$M_1 \eta_1$	$M_2 (I - \alpha_2)^{-1} \delta_2$	$M_2 (I - \alpha_2)^{-1} \eta_2$

are those which have the largest effect on x_1 and/or have large absolute values and thus correspond to slow modes). This situation can always be realized by an appropriate rearrangement of the elements of α and the columns of M. This partitioning of α justifies the simplifying assumption that z_2 reacts instantaneously to changes in u or d. Once this approximation is made, the expressions for φ_R, Δ_R, and θ_R in Table 1 can be derived (Wilson et al., 1972, 1973). In Table 1, Δ_1, φ_1 and φ_2 are partitions of Δ and φ, and V_3, and V_4 are partitions of V where $V \equiv M^{-1}$ (Wilson et al., 1972, 1974). An important advantage of Marshall's method is that the reduced-order model has the same steady state as the high-order model for step disturbances.

As noted by Graham (1968), one of Chidambara's proposed methods (Chidambara and Davison, 1967b) is equivalent to Marshall's method in the sense that the resulting reduced-order models are the same.

Davison's Method

A second basic model reduction technique has been developed by Davison (1966) and Nicholson (1964). As shown by Wilson (1974), these two formulations generate the same reduced-order model. Since this approach is referred to in the literature as *Davison's method,* the same designation will be used in this paper.

The basis of this approach is to derive a reduced-order model in which the modes are excited in the same relative proportions as they are in the high order model. This objective is achieved but at the expense of a steady state error between the high-order and low-order model responses to a step disturbance. The resulting reduced-order model is shown in Table 1. Several modifications of Davison's method have been reported but many of these are equivalent, as demonstrated in Appendix I.

Davison (1968b) suggested a revision to his earlier method (Davison, 1966) which would ensure steady state agreement for single-input systems. Here each element of x_1 is multiplied by the ratio of the desired and actual steady state values and it is assumed that both φ and φ_R are nonsingular.

A second modification of Davison's method has been presented in different forms by both Chidambara and Davison (1967b). Chidambara later noted the equivalence of these two formulations (Chidambara and Davison, 1967c). The reduced-order model is shown in Table 1 as the revised Davison method. Fossard (1970) and Graham (1968) have also proposed model reduction techniques, but their approaches yield reduced-order models which are identical to those of the previously reported revision of Davison's method (Wilson, 1974). Apparently, the equivalence of these methods has not been previously reported and hence is included in the Appendix. The resulting reduced-order model can be written as

$$x_R \equiv x_l + E_R\, u + F_R\, d \qquad (14)$$

where the l-dimensional vector x_l is calculated from

$$x_l(j+1) = \varphi_R\, x_l(j) + \Delta_R\, u(j) + \theta_R\, d(j) \qquad (15)$$

As shown in Table 1, the φ_R, θ_R and Δ_R matrices in Equation (15) are the same as in Davison's original method.

CONTROL LAW REDUCTION

In many control problems a satisfactory state feedback controller for the high-order model can be designed but is not practical for actual implementation due to the unavailability of a significant number of state variables. The model reduction approach described in the previous section could be used, but it does make use of all the available information, namely, the high-order control law. An alternative strategy is to use both the high-order controller and the high-order model to design the reduced-order controller. This approach will be referred to as control law reduction and consists of steps 5 and 8 in Figure 1. Control law reduction based on a modal analysis of the high-order model has been developed by the authors (Wilson et al., 1973) and is briefly summarized below.

The basis of the method is to use a modal analysis of the high-order model in Equation (3) to derive an approximate expression for x_2 in terms of x_1, u, and d. The resulting expression for x_2 is approximate because of the simplifying assumption that the fast modes of the high-order model (corresponding to z_2) react instantaneously (that is, the same assumption that is made in Marshall's model reduction method). Substitution of this expression into Equation (5) and subsequent rearrangement yields the reduced-order controller of Equation (7) with the following control matrices (Wilson et al., 1973):

$$\begin{aligned}
K_R^{FB} &= K^u\,(K_1^{FB} - K_2^{FB}\, V_4^{-1}\, V_3)\\
K_R^{I} &= K^u\, K^I\\
K_R^{FF} &= K^u\,(K^{FF} + K_2^{FB}\, V_4^{-1}\,(I-\alpha_2)^{-1}\,\eta_2) \qquad (16)\\
K_R^{SP} &= K^u\, K^{SP}\\
K^u &= (I - K_2^{FB}\, V_4^{-1}\,(I-\alpha_2)^{-1}\,\delta_2)^{-1}
\end{aligned}$$

where

$$V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} = M^{-1} \qquad (17)$$

APPLICATION TO A PILOT PLANT EVAPORATOR

The double effect evaporator used in this work is located in the Department of Chemical Engineering at the University of Alberta and has been described in previous publications (for example, Newell et al., 1972a, b, c). However, for convenience, a schematic diagram, a list of the process variables, and the normal steady state operating conditions are included in Appendix II.*

The control objective is to maintain the product concentration C2 constant in spite of variations in load variables such as feedflow rate F. It is also necessary to control the liquid holdups in the two effects, W1 and W2, within practical limits. Control is maintained by manipulating the inlet stream flow S, the liquid flow rate (bottoms) B1, from the first effect, and the product flow rate B2. The evaporator feed and product handling units and the computer control system form an important part of the pilot plant, but their description is not necessary for the purpose of this paper.

* Appendix II is available from the authors upon request.

Several of the evaporator models developed over the last eight years are discussed and compared by Newell and Fisher (1972b). The most rigorous and accurate model that has been used consists of ten nonlinear, first-order ordinary differential equations. This model was linearized and put into the standard linear state-space form defined by Equations (3) and (4) and used as the starting point for this work.

DISCUSSION OF RESULTS

The objective of this investigation was to start with a high-order theoretical model and develop the lowest-order, practical, multivariable, feedback controller that would give satisfactory control of the existing pilot plant evaporator. As discussed in the previous sections, the work divided into two main areas: model reduction (paths 4 and 7 in Figure 1) and control law reduction (paths 5 and 8). However, the task of evaluating the different design methods via simulated and/or experimental runs was complicated by the large number of different factors involved. The most important factors are summarized below:

1. Model Order: The lower-order models referred to in this paper were derived from the linear tenth-order, state-space model using the discrete equivalent (Wilson et al., 1972) of Marshall's model reduction technique (Marshall, 1966). The 10th order plus 5th, 4th, and 3rd-order models were evaluated in open-loop and closed-loop runs.
2. Reduction Techniques: The model reduction and control law reduction steps were carried out using several different techniques. The modal approaches are presented here and the least squares results are described elsewhere (Wilson, 1974).
3. Design Strategy: The major emphasis was placed on comparing model reduction versus control law reduction, that is, paths 4 and 7 versus 5 and 8. For ease of reference the optimal control law for a β order model is designated βth, for example, for a third-order system 3OPT. When the optimal control law for the tenth-order system is reduced to a third-order controller using the procedure described previously, the resulting controller is designated as 3RED10.
4. Type of Control Law: Multivariable feedback controllers were designed for the tenth-order and the reduced-order models using the same optimal, linear, quadratic formulation used in previous work (for example, Newell et al., 1972a). The weighting factors in the quadratic performance index (that is, the elements of the **Q** and **R** matrices) were selected based on the previous results and the same values were used for all runs discussed in this paper. The basic control mode was multivariable proportional feedback control, but runs were also done to evaluate the effect of adding integral feedback, proportional feedforward, and setpoint (servo) control. Note that any other design procedure for multivariable state feedback controllers could be substituted for the optimal control approach used in this work.
5. Type of Disturbance: Since the reduced-order models only approximate the behavior of the actual nonlinear process, the control schemes were evaluated for disturbances in different load variables, for nonzero initial conditions, and for setpoint changes.
6. Multivariable Aspects: The evaporator is a multi-input, multi-output system and hence performance evaluation is complicated. For example, techniques that produce an improved response for one state variable could result in poorer control of other variables and/or excessive manipulation of input variable(s). No single quantitative criterion was found that would satisfactorily characterize the performance of all the runs. Therefore, all of the variables were plotted (Wilson, 1974) and the evaluation made subjectively. In this paper, the most important state variable C2 is plotted in each figure and typical responses of a second state variable and/or manipulated variables are included in some figures.

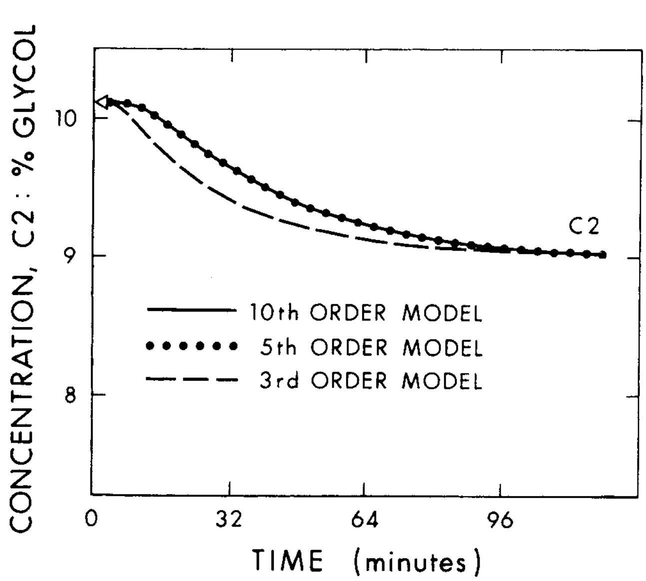

Fig. 2. Simulated response of different order evaporator models to a +20% step in feedflow to show the effect of model order. (Arrowheads indicate initial steady state values.)

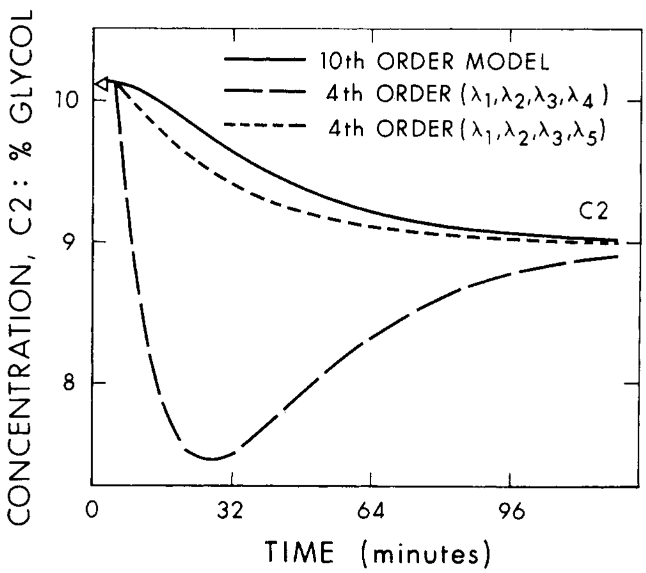

Fig. 3. Simulated responses of different evaporator models to a +20% step in feedflow to show the effect of retaining different eigenvalues in the reduced-order model.

The coefficient matrices for the tenth-order model, the tenth-order optimal controller, and the third-order model from the paper by Wilson et al. (1973) are reproduced in Appendix 2. Further details are available from the thesis by Wilson (1974). The following discussion describes some of the results obtained and the conclusions reached.

Effect of Model Order

The accuracy of a state-space model generally decreases as the model order is reduced. For example, Figure 2 shows that the product concentration response of the fifth-order evaporator model was essentially identical to that of the original tenth-order model, but the third-order model exhibits a significant error. The models were reduced using modal analysis techniques, retaining the largest system

TABLE 2A. CONTROL MATRICES FOR PROPORTIONAL-PLUS-INTEGRAL CONTROL

Design method	K_R^{FB}			K_R^{I}		
3RED10	2.977	0.0937	−5.408	0.4117	0.02088	−0.4861
	5.066	−1.246	8.10	0.8398	−0.2887	0.8159
	5.686	12.29	11.58	0.8559	1.945	1.068
30PT	5.49	−0.1903	−12.00	0.9893	−0.05066	−1.175
	6.429	−1.386	4.487	1.156	−0.3255	0.4373
	5.519	12.26	11.81	0.8254	1.935	1.090

TABLE 2B. CONTROL MATRICES FOR PROPORTIONAL FEEDBACK, FEEDFORWARD AND SETPOINT CONTROL

Design method	K_R^{FB}			K_R^{FF}			K_R^{SP}		
3RED10	2.467	0.02163	−4.705	1.238	−0.5639	−0.4138	−2.468	−0.0212	5.276
	4.288	−1.340	8.885	0.9815	0.2177	−0.00123	−4.289	1.340	−9.100
	4.128	9.760	9.528	0.9978	0.9877	−0.00140	−4.128	−9.763	−10.52
30PT	4.904	−0.4013	−11.92	1.238	−0.5583	−0.4128	−4.904	0.4017	12.47
	5.784	−1.600	4.425	0.9832	0.2231	0.00055	−5.784	1.599	−4.648
	4.093	9.685	9.357	0.9983	0.9937	−0.00122	−4.094	−9.686	−10.35

eigenvalues and with $\mathbf{x}_l = [W1, C1, H1, W2, C2]^T$ and $[W1, W2, C2]^T$, respectively.

The results in Figure 3 show that the assumptions and/or design decisions made during the model reduction step can be more important than how much the model order is reduced, that is, than the final model order. The two fourth-order models used in Figure 3 have the same state vector, $\mathbf{x}_l = [W1, H1, W2, C2]^T$, but differ in the subset of eigenvalues of the tenth-order model that is retained in the reduced model [compare Equation (9) and related discussion]. In this run the objective was to obtain a reduced model that would satisfactorily represent the open-loop response to a step disturbance. Thus it would appear logical to retain the dominant (largest in absolute value) eigenvalues of the original system in the reduced model. However, when the four largest eigenvalues of φ were retained in φ_R, the model response was very poor as illustrated by the C2 response in Figure 3 (long dashes). When the fourth largest eigenvalue (0.9292) was replaced by the fifth largest (0.7354), the response was much better as shown by the curve of short dashes. This difference can be explained (or predicted a priori) by an examination of the modal matrix for the tenth-order model which shows that element H1 of the state vector depends significantly on the fifth eigenvalue but not on the four largest ones (Wilson et al., 1973). In other words, since $\mathbf{x} = \mathbf{M}\ \mathbf{z}$, it is important when considering the response of the ith element of $\mathbf{x}$ to inspect the ith row of $\mathbf{M}$ as well as at the dynamics of all the elements of $\mathbf{z}$ (which are characterized by the diagonal elements of $\boldsymbol{\alpha}$ which in turn are equal to the eigenvalues of φ).

Model Reduction Techniques

The modal methods applied to the reduction of the evaporator model are those summarized in Table 1.

Figure 4 compares the simulated time domain response to a +20% step change in feedflow using three third-order and the tenth-order open-loop models. The third-order model calculated using Davison's method has considerable steady state error. [It should be pointed out, however, that the fifth-order model calculated using Davison's method had negligible steady state error (Wilson, 1974).] Figure 4 also shows that the third-order models calculated using Marshall's method and the revised Davison method are very similar, lead the tenth-order model slightly, and result in zero steady state error. In order to have zero initial error

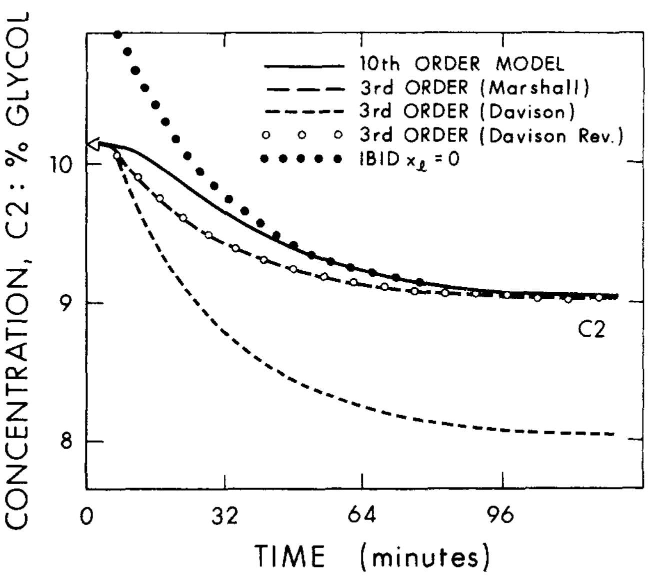

Fig. 4. Simulated responses of the tenth-order and reduced-order evaporator models to a 20% step in feedflow to compare different modal analysis techniques.

using the revised Davison method, the initial values of the elements of $\mathbf{x}_l$ were calculated as follows:

$$\mathbf{x}_l(0) = \mathbf{x}_1(0) - \mathbf{E}_R\ \mathbf{u}(0) - \mathbf{F}_R\ \mathbf{d}(0) \qquad (18)$$

If $\mathbf{x}_R(0) \neq \mathbf{x}_1(0)$ is used, as proposed by Graham (1968), then there is a significant error in the initial value of $\mathbf{x}_R$ but, as illustrated by the dotted C2 response in Figure 4, the response converges rapidly to that of the tenth-order model.

Effect of the Design Sequence

In the closed-loop control runs, the main objective was to compare the low-order controllers designed using the control law reduction and model reduction approaches. In each design the modal analysis of Marshall was employed to simplify either the controller or the model. Most of the closed-loop experimental runs presented in this paper used third-order controllers with $\mathbf{x}_l = [W1, W2, C2]^T$. The controller matrices are given in Table 2. Typical results using fifth-order controllers are presented in Figures 8 and 9 and in the literature (Newell et al., 1972a, c).

Figure 5 shows typical experimental responses to 20% changes in feedflow when using a proportional feedback plus feedforward controller designed using the control law

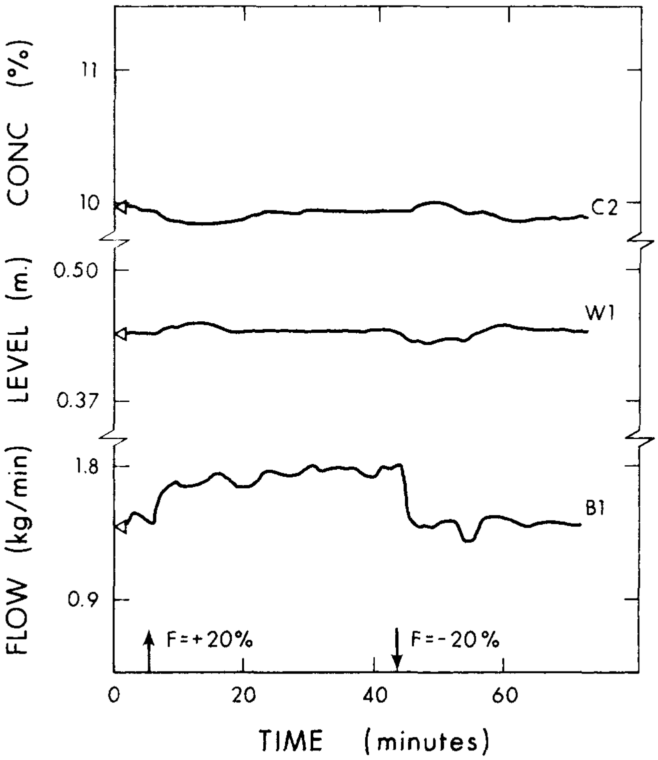

Fig. 5. Experimental evaporator response to 20% step changes in feedflow with a 3RED10 controller incorporating FB+FF modes. (Arrows on the time axis denote a ±20% disturbance in feedflow.)

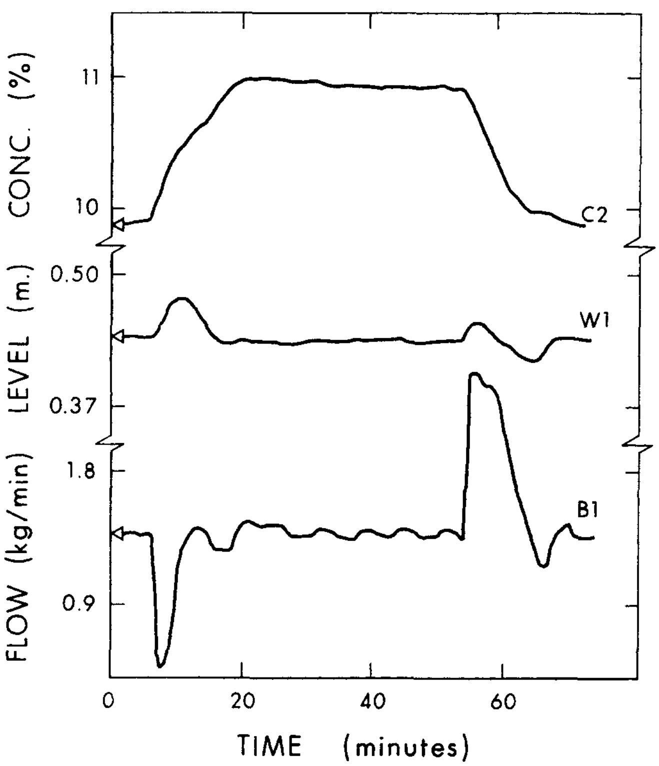

Fig. 6. Experimental evaporator response to 10% step changes in the product concentration (C2) setpoint with a 3RED10 controller incorporating FB+SP modes.

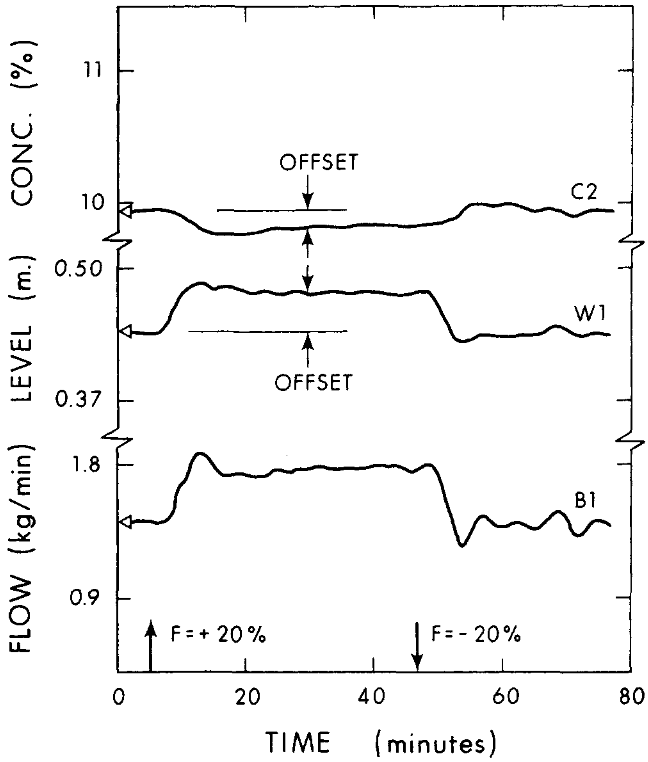

Fig. 7. Experimental evaporator response to 20% step changes in feedflow with a 3RED10 controller incorporating only multivariable proportional feedback control. (Note the offset in C2 and W1 compared to Figure 5.)

reduction approach. This 3RED10 controller produced slightly steadier control with less oscillatory responses than a similar 30PT controller designed on the basis of a third-order model (results not shown—see Wilson, 1974, Figure 6.11). Note that some of the corresponding elements in the controller matrices presented in Table 2 differ by over a factor of two. A comparison of Figures 5 and 7 shows that the addition of proportional feedforward control eliminated the steady state offsets in the state variables.

The results in Figure 6 were obtained using a 3RED10 controller designed using the same approach as for Figure 5. However, in this case the controller included proportional feedback plus setpoint control modes and the disturbances were 10% changes in the setpoint of the product concentration C2. Note that since the control law is

$$\mathbf{u} = \mathbf{K_R}^{FB}\, \mathbf{x}_1 + \mathbf{K_R}^{SP}\, \mathbf{y}^{SP} \tag{19}$$

and the variables represent perturbations about the initial steady state, then when $\mathbf{y}^{SP} = \mathbf{0}$ (as in Figure 6 for $t >$ 55 min.) only the proportional feedback control mode is active. Thus the responses represent a return from nonzero initial conditions. (Figure 7 presents the comparable results for a disturbance in feed flow.) The results in Figure 6 showed significantly less overshoot in C2 and smoother, less oscillatory responses than those obtained using a comparable 30PT controller designed using the model reduction approach. The latter results are available elsewhere (see Wilson 1974, Figure 6.15).

The results in Figure 7 were also obtained by control law reduction and represent the response to 20% step changes in feedflow when using proportional feedback control only. The responses of both the state and manipulated variables were smoother and less oscillatory than any of the multiple mode controllers. The control matrix $\mathbf{K_R}^{FB}$ is given in Table 2. Note that the steady state offsets which are expected when proportional control is used alone are evident in both C2 and W1. These offsets were eliminated when the second feed flow disturbance brought the system back to the original steady state values. The comparable

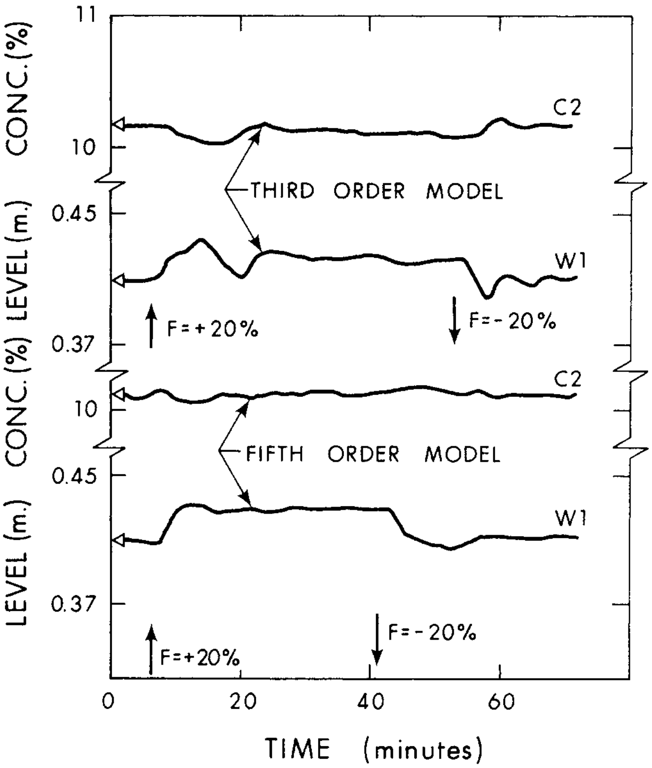

Fig. 8. Experimental evaporator responses to 20% step changes in feedflow to compare "30PT" (top) and "50PT" (bottom) proportional feedback controllers.

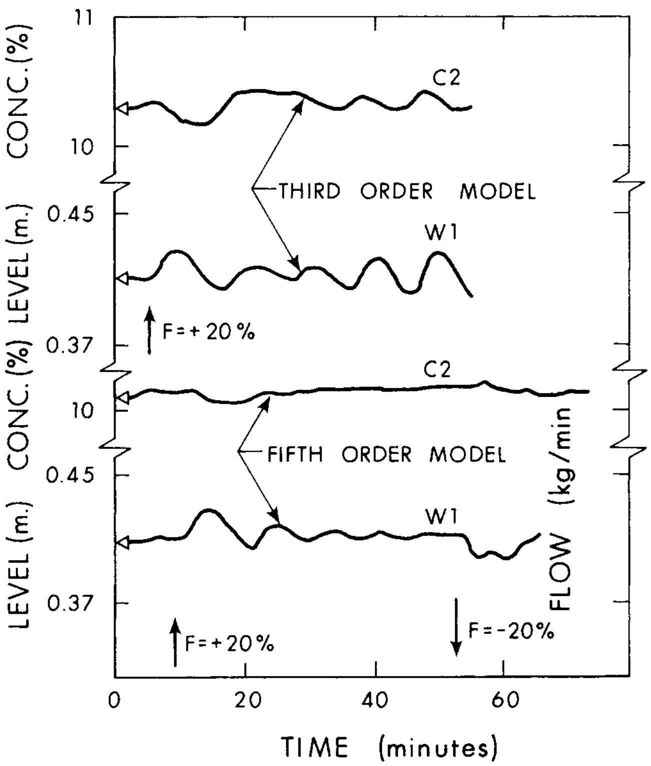

Fig. 9. As per Figure 8 but using proportional plus integral feedback modes. Note that the offsets apparent in Figure 8 are eliminated, but the responses are more oscillatory.

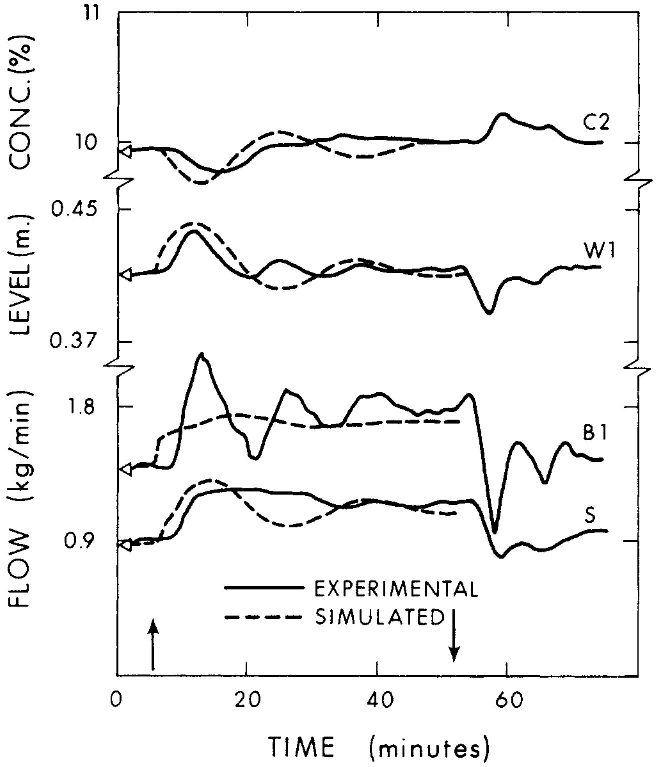

Fig. 10. Comparison of simulated versus experimental evaporator responses to 20% step changes in feedflow with a 3RED10 incorporating FB+I modes.

30PT controller had generally higher feedback gains which produced slightly more oscillatory responses and smaller offsets as shown by the curves in the top half of Figure 8.

Although they are included for other purposes, the curves in the top half of Figures 9 and 10 provide another comparison of the two approaches for developing low-order controllers. Figure 10 uses a 3RED10 proportional plus integral feedback controller developed by control law reduction. Figure 9 is the comparable 30PT controller. Both responses are to 20% disturbances in feedflow. Note that the responses in Figure 9 are more oscillatory and in fact the response of the liquid holdup W1 is unstable.

In summary, in the evaporator application the control law reduction approach tended to produce better low-order controllers since the controller gains were generally lower and the transient responses were less oscillatory with smoother changes in the manipulated variables.

Selection of Controller Order

The results plotted in the bottom half of Figures 8 and 9 show the response of the evaporator when using fifth-order optimal controllers and are directly comparable to previous work, for example, Newell et al., (1972a). Comparison of these results with those in the top half of Figures 8, 9, and 10 shows that, as expected, controllers based on a fifth-order model are better than the 3rd-order controllers designed using model reduction or control law reduction. However, there are compensating factors; for example, with the fifth-order controller, it is necessary to measure or estimate the first effect concentration C1 and the implementation of the controller is more difficult. The comparison of transient responses in Figure 10 for a 3RED10 controller shows that there are significant differences between simulated and experimental results. For example, the time delay evident in the experimental results is not adequately described by the process model.

Unfortunately there is no way of accurately predicting the effect of model reduction or control law reduction on the actual closed-loop process responses. Guarantees of optimality, stability, integrity, etc., which may apply to the high-order model and controller are certainly lost when simplified controllers are applied to the actual process. Thus an iterative design approach followed by experimental evaluation and tuning may be required in complex applications. However, simulation is still an effective means for preliminary evaluations and comparison of alternatives.

NOTATION

$\mathbf{A}$ = coefficient matrix in Equation (1)
$\mathbf{B}$ = coefficient matrix in Equation (1)
$\mathbf{C}$ = coefficient matrix in Equation (2)
$\mathbf{D}$ = coefficient matrix in Equation (1)
$\mathbf{d}$ = disturbance vector, $q \times 1$
$\mathbf{E}$ = coefficient matrix
$\mathbf{F}$ = coefficient matrix
$\mathbf{I}$ = identity matrix
j = time interval counter
$\mathbf{K}$ = control matrix
$\mathbf{K}_1$, $\mathbf{K}_2$ = partitions of $\mathbf{K}$
l = dimension of $\mathbf{x}_1$
$\mathbf{M}$ = modal matrix
$\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, $\mathbf{M}_4$ = partitions of $\mathbf{M}$
m = dimension of control vector
N = number of data points
n = dimension of state vector
p = dimension of output vector
q = dimension of disturbance vector
t = time
$\mathbf{u}$ = control vector, $m \times 1$
$\mathbf{V}$ = $\mathbf{M}^{-1}$
$\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ = partitions of $\mathbf{V}$
$\mathbf{x}$ = state vector, $n \times 1$
$\mathbf{x}_1$, $\mathbf{x}_2$ = partitions of $\mathbf{x}$
$\mathbf{y}$ = output vector, $p \times 1$
$\mathbf{z}$ = canonical state vector, $n \times 1$
$\mathbf{z}_1$, $\mathbf{z}_2$ = partitions of $\mathbf{z}$

Greek Letters

$\boldsymbol{\alpha}$ = $\mathbf{M}^{-1} \boldsymbol{\varphi} \mathbf{M}$
$\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ = partitions of $\boldsymbol{\alpha}$
$\boldsymbol{\Delta}$ = coefficient matrix in Equation (3)
$\boldsymbol{\Delta}_1$ = partition of $\boldsymbol{\Delta}$
$\boldsymbol{\delta}$ = $\mathbf{M}^{-1} \boldsymbol{\Delta}$
$\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, = partitions of $\boldsymbol{\delta}$
$\boldsymbol{\eta}$ = $\mathbf{M}^{-1} \boldsymbol{\theta}$
$\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ = partitions of $\boldsymbol{\eta}$
$\boldsymbol{\theta}$ = coefficient matrix in Equation (3)
$\boldsymbol{\varphi}$ = coefficient matrix in Equation (3)
$\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ = partitions of $\boldsymbol{\varphi}$

Subscript

R = reduced-order

Superscripts or Abbreviatioss

FB = proportional (state) feedback control
FF = proportional feedforward control
I = integral feedback control
SP = setpoint control
T = transpose
αOPT = a control law of order α generated using the optimal-linear quadratic formulation and a model of order α, for example, 3OPT
αREDβ = a control law of order α generated by reducing an optimal control law of order β, for example, 3RED10
-1 = matrix inverse
$\cdot$ = time derivative

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APPENDIX I

Many of the modal approaches for model reduction that have been reported in the literature as independent methods are in fact equivalent since they produce identical reduced-order models. In particular,

1. The formulations of Davison (1966) and Nicholson (1964) are equivalent.
2. Marshall's method (1966) and one of Chidambara's proposed methods (Chidambara and Davison, 1967b) are equivalent, as previously noted by Graham (1968).
3. The approaches of Fossard (1970) and Graham (1968) are equivalent to a revision of Davison's method derived by Chidambara and Davison (1967b).

Proofs that the resulting reduced-order models are equivalent were developed as part of this investigation and are available elsewhere (Wilson, 1974).

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Developing and Fully Developed Velocity Profiles for Inelastic Power Law Fluids in an Annulus

Developing and fully developed velocity profiles with respect to position for laminar flow of inelastic fluids in an annulus were measured using streak photography. The polymer solutions used in the investigation were found to exhibit power law behavior under the present experimental conditions. The experimental results substantiated published theoretical solutions for developing and fully developed flow of power-law fluids in annuli, and also indicated good agreement between the measured and predicted entrance lengths. The dimensionless entrance length was found to increase with increasing flow behavior index for the inelastic power-law fluids.

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SCOPE

Developing and fully developed laminar flow of non-Newtonian fluids in annuli is of importance in many industrial applications such as in heat transfer, in plastic extrusion, in mixing of viscous liquids, and in well bore fluid circulation. Experimental work in this area of research, particularly the entry flow problem, is almost nonexistent. When a fluid enters an annulus through a contraction, it undergoes a change in the flow pattern from an initial velocity profile at the inlet to a fully developed profile at a certain distance downstream. Associated with the developing field is an increase in kinetic energy and in viscous friction, which results in a higher pressure drop in the entrance region. Hence, quantitative information on entrance length, defined as the distance from the inlet to the point of fully developed flow, is of importance to the design or process engineer. Although most highly viscous non-Newtonian fluids are also elastic in nature, an understanding of the inelastic flow behavior in an annular entry is essential before the more complicated viscoelastic problem can be tackled. In the present study experimental measurements of developing and fully developed velocity profiles in an annulus are presented for some inelastic polymer solutions. The measured profiles are used to compare and substantiate published theoretical solutions. The entrance lengths in the annulus are established from the developing velocity profiles.

CONCLUSIONS AND SIGNIFICANCE

The significance of the present work lies in the accurate photographic measurements of velocity distributions for the laminar flow of non-Newtonian fluids in an annulus. The theoretical fully developed velocity profiles obtained from the solution of Fredrickson and Bird (1958) for the flow of power-law fluids in annuli were substantiated for the first time with the present experimental measurements. The measured developing velocity profiles for inelastic fluids in the entrance region showed excellent agreement with the momentum-energy integral solution published previously by the authors (1973). Therefore, a quantitative estimate of the entrance length in an annulus can

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